MAS224, Actuarial Mathematics: Life Tables (2nd part of Lecture 13)

Tables containing estimates of the values of life-table functions for exact ages $x = 0, 1, 2, \dots$ are called the life tables. The first life table was published in 1693 by Edmund Halley, who based his table on the register of births and deaths of the city of Breslau (now Wrocław).

Edmund Halley, (b. 1656 – d. 1742), English astronomer and mathematician, was the first person to calculate the orbit of a comet which known now as Halley's Comet. (Also published the 1st meteorological chart (contained the distribution of prevailing winds over oceans) and the 1st magnetic charts of the Atlantic and Pacific areas.)

In this course we shall be using two life tables: *English Life Table No. 12 – Males* and *A1967-70*.

Life tables are not constructed by observing l_0 newborns until the last survivor dies (one would need to wait too long!). Instead, life tables are based on estimates of probabilities of death, given survival to various ages derived from the mortality experienced by the entire population (or its groups) over three consequitive years. English Life Table No. 12 is constructed on the basis of the mortality experienced by the entire male population of England in 1960, 1961 and 1962. The A1967-70 table is based on the experience, within these years, of lives assured by UK life assurance companies.

*Worked Example T*oy example of constructing a life table

Suppose you know q_x for all integer x (or, equivalently, $p_x = 1 - q_x$. Then you can construct a life table using $l_{x+1} = p_x l_x$. For instance, consider an animal population with a lifespan of 5 years (i.e. the animals live at most 5 years) with

$$\begin{array}{ll}
p_0 = 0.5 & = P(T(0) > 1) = P(X > 1) \\
p_1 = 0.4 & = P(T(1) > 1) = P(X > 2|X > 1) \\
p_2 = 0.3 & = P(T(2) > 1) = P(X > 3|X > 2) \\
p_3 = 0.2 & = P(T(3) > 1) = P(X > 4|X > 3) \\
p_4 = 0.1 & = P(T(4) > 1) = P(X > 5|X > 4) \\
p_5 = 0 & = P(T(5) > 1) = P(X > 6|X > 5)
\end{array}$$

Set the radix: $l_0 = 10000$, say. Then (as $l_{x+1} = l_x \times p_x$)

$l_1 = l_0 p_0 = 10000 \times 0.5 = 5000$, $l_2 = l_1 p_1 = 5000 \times 0.4 = 2000$, etc.;

$d_0 = l_0 - l_1 = 10000 - 5000 = 5000$; $d_1 = l_1 - l_2 = 5000 - 2000 = 3000$, etc.;

$\mathrm{e}_0 = \frac{l_1+l_2+l_3+l_4+l_5}{l_0} = \frac{5000+2000+600+120+12}{10000} = 0.7732$, hence $\mathring{\mathrm{e}}_0 \approx \mathrm{e}_0 + \frac{1}{2} = 1.2732$

$\mathrm{e}_1 = \frac{l_2+l_3+l_4+l_5}{l_1} = \frac{2000+600+120+12}{5000} = 0.5464$, hence $\mathring{\mathrm{e}}_1 \approx \mathrm{e}_1 + \frac{1}{2} = 1.0464$, etc.

Turn Over ...

x	l_x	d_x	p_x	q_x	$\mathring{e}_x$	x
0	10000	5000	0.5	0.5	1.2732	0
1	5000	3000	0.4	0.6	1.0464	1
2	2000	1400	0.3	0.7	0.8660	2
3	600	480	0.2	0.8	0.7200	3
4	120	108	0.1	0.9	0.6000	4
5	12	12	0	1	0.5000	5

Worked Example .

Assume mortality of the English Life Table No. 12 – Males.

(a) Find the probability for a newborn to survive to age 18.

(b) Find the probability for a man exact age 25 to survive to age 60.

(c) Find the probability for a man exact age 25 to die within the next 35 years of life.

(d) Find the probability for a man exact age 25 to die between ages 60 and 70.

(e) Estimate the number of the survivors to age 60 from a group of 9000 newborns.

Solution.

Recall X is the time-until-death for a newborn. $T(x)$ is the future lifetime of (x), i.e. a person aged x, and $T(x) = X - x$ given $X > x$. Therefore:-

(a) Need to find $P(T(0)>18)$ or, equivalently, $P(X>18)$.

$$P(T(0)>18)=P(X>18)={}_{18}p_0=\frac{l_{18}}{l_0}=\frac{96514}{100000}=0.96514.$$

(b) Need to find $P(T(25)>35)$ or, equivalently, $P(X>35+25|X>25)$.

$$P(T(25)>35)=P(X>60|X>25)={}_{35}p_{25}=\frac{l_{60}}{l_{25}}=\frac{78924}{95753}=0.8242.$$

(c) Need to find $P(T(25)\leq 35)$ or, equivalently, $P(25<X\leq 25+35|X>25)$.

$$P(T(25)\leq 35)=1-P(T(25)>35)=1-{}_{35}p_{25}=1-\frac{l_{60}}{l_{25}}=1-\frac{78924}{95753}=0.1758.$$

or

$$P(T(25)\leq 35)=P(25<X\leq 60|X>25)={}_{35}q_{25}=\frac{l_{25}-l_{60}}{l_{25}}=\frac{95753-78924}{95753}=0.1758.$$

(d) Need to find $P(60-25<T(25)\leq 70-25)$ or, equivalently, $P(60<X\leq 70|X>25)$.

$$P(35<T(25)\leq 45)=P(60<X\leq 70|X>25)={}_{35|10}q_{25}=\frac{l_{60}-l_{70}}{l_{25}}=\frac{78924-54806}{95753}=0.25188.$$

(d) We estimate this number by the expected number of survivors to age 60, which is given by

$$9000\times P(X>60)=9000\times {}_{60}p_0=9000\frac{l_{60}}{l_0}=\frac{9000}{100000}l_{60}=\frac{9000}{10000}78924=7100.$$

The Curve of Deaths

By the definition of ${}_t d_x$, ${}_u d_x$ is the expected number of deaths in the age interval between x and $x + u$ in a group of l_0 newborns.

$$\begin{aligned} {}_u d_x = l_x - l_{x+u} &= l_0 s(x) - l_0 s(x+u) \\ &\approx -l_0 \Big[\frac{d}{dx} s(x)\Big] \times u \qquad \text{[for small } u\text{]} \\ &= l_x \mu(x) u. \end{aligned}$$

Therefore, for small u, the expected number of deaths in the age interval $(x, x + u)$ is approximately $l_x\, \mu(x)\, u$. Hence, $l_x\, \mu(x)$ can be interpreted as the rate of (expected) deaths at age x per l_0 newborns. Because of this interpretation, the plot of $l_x\, \mu(x)$ against x, is often called the *curve of deaths*.

Examine peculiarities of the curve of deaths based on the mortality of the ELT-12-Males.